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Note

Numerical Solution of the Shallow Water Equations

The hyperbolic quasi-linear system of equations governing the flow of an ideal incompressible fluid in a gravitational field is known as the shallow water equations. In the solution of these equations discontinuities arise when bores or hydraulic jumps are present. An example of such a phenomenon is provided by the problem of the breaking of a dam [1], and is also analogous to the Riemann problem of one-dimensional gas dynamics [2]. Several numerical methods have been proposed for the numerical solution of these equations [3–5]. In the most recent of these methods [5], a langrangian method was developed for the shallow water equations based on a Voronoi mesh. Although the method gave good agreement with the theoretical solution for the rarefaction and shock waves positions and speeds, it showed an unphysical overshoot behind the shock and unphysical oscillations in the constant state near the rarefaction wave [5]. In Ref. [4], more accurate results have been presented using a Random Choice Method.

In view of the importance of this problem, and of the extensive literature devoted to the subject, it is purpose of the present note to apply a much simpler algorithm based on shape preserving splines [6, 7] for the solution of this problem.

Shape-preserving interpolants based on Bernstein polynomials have been given increasing attention in computational geometry. Bézier [8] has developed and described the mathematical basis for a successful system for the design of curves and surfaces. An alternative development, in which the Bézier method emerges as an application to the Bernstein polynomials approximation, and the extension of the Bézier technique to splines have been described in Refs. [9, 10]. The application of shape preserving splines based on Bernstein polynomials for the numerical solutions of differential equations has been presented in [6, 7]. It is the purpose of the present note to apply the algorithm developed in [6, 7] for the solution of the shallow water equations. The details of this algorithm have been given in [6, 7] and will not be repeated here.

We are dealing essentially with the shallow water equations in one space dimension. These equations are given by [1, 4, 5]:

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0, \tag{2}$$

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where $g = 9.8066 \text{ m/sec}^2$, h and u are the depth and the fluid velocity, respectively. These equations are solved for the dam-breaking problem [1] with the initial conditions

$$u(x, 0) = 0.2667 \text{ m/sec} \qquad x < 0 \tag{3a}$$

$$= 1.6 \text{ m/sec}, \qquad x > 0, \tag{3b}$$

$$h(x, 0) = 10.8 \text{ m} \qquad x < 0 \tag{4a}$$

$$= 1.8 \text{ m}, \qquad x > 0. \tag{4b}$$

The initial conditions are shown in Figs. 1a and 2a for $h(x, 0)$ and $u(x, 0)$ respectively.

At any time t, the exact solution consists of four regions, where the height of the fluid $h(x, t)$ is given by

$$h(x, t) = 1.8 \text{ m} \qquad x > 10.7t \tag{5a}$$

$$= 4.716 \text{ m} \qquad 0.45t \leqslant x \leqslant 10.7t \tag{5b}$$

$$= \frac{1}{88.2}\left(20.8401 - \frac{x}{t}\right)^2 \text{m} \qquad -10.02t \leqslant x \leqslant 0.45t \tag{5c}$$

$$= 10.8 \text{ m} \qquad x \leqslant -10.02t. \tag{5d}$$

So a shock wave propagates in the positive direction with a velocity 10.7 m/sec [1], and a rarefaction wave propagates in the negative direction with a velocity -10.02 m/sec.

Equations (1) and (2) are first modified using the Riemann invariants, which are constant quantities along characteristic lines. We obtain the following system of equations:

$$\left(\frac{\partial}{\partial t} + v_+ \frac{\partial}{\partial x}\right) U_+ = 0, \tag{6}$$

$$\left(\frac{\partial}{\partial t} + v_- \frac{\partial}{\partial x}\right) U_- = 0, \tag{7}$$

where

$$c = (gh)^{1/2} \tag{8}$$

and

$$v_+ = u + c, \tag{9}$$

$$v_- = u - c, \tag{10}$$

$$U_+ = u + 2c, \tag{11}$$

$$U_- = u - 2c. \tag{12}$$

The numerical scheme to solve Eqs. (6) and (7) is the following. We use a two-

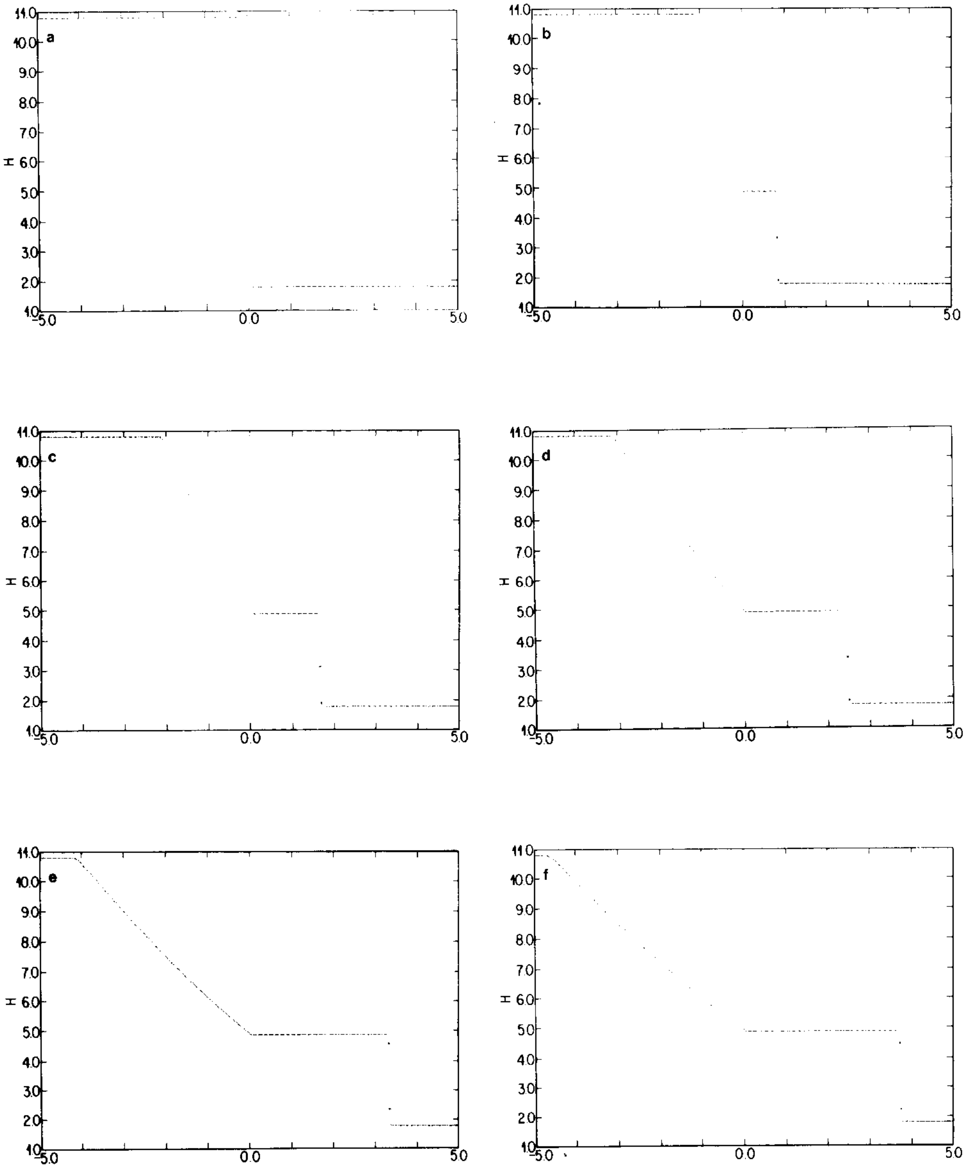

FIG. 1. Profile of the height $h(x, t)$ as a function of x at (a) $t=0$, (b) $t=100$, (c) $t=200$, (d) $t=300$, (e) $t=400$, (f) $t=450$ sec. The distance x is in km.

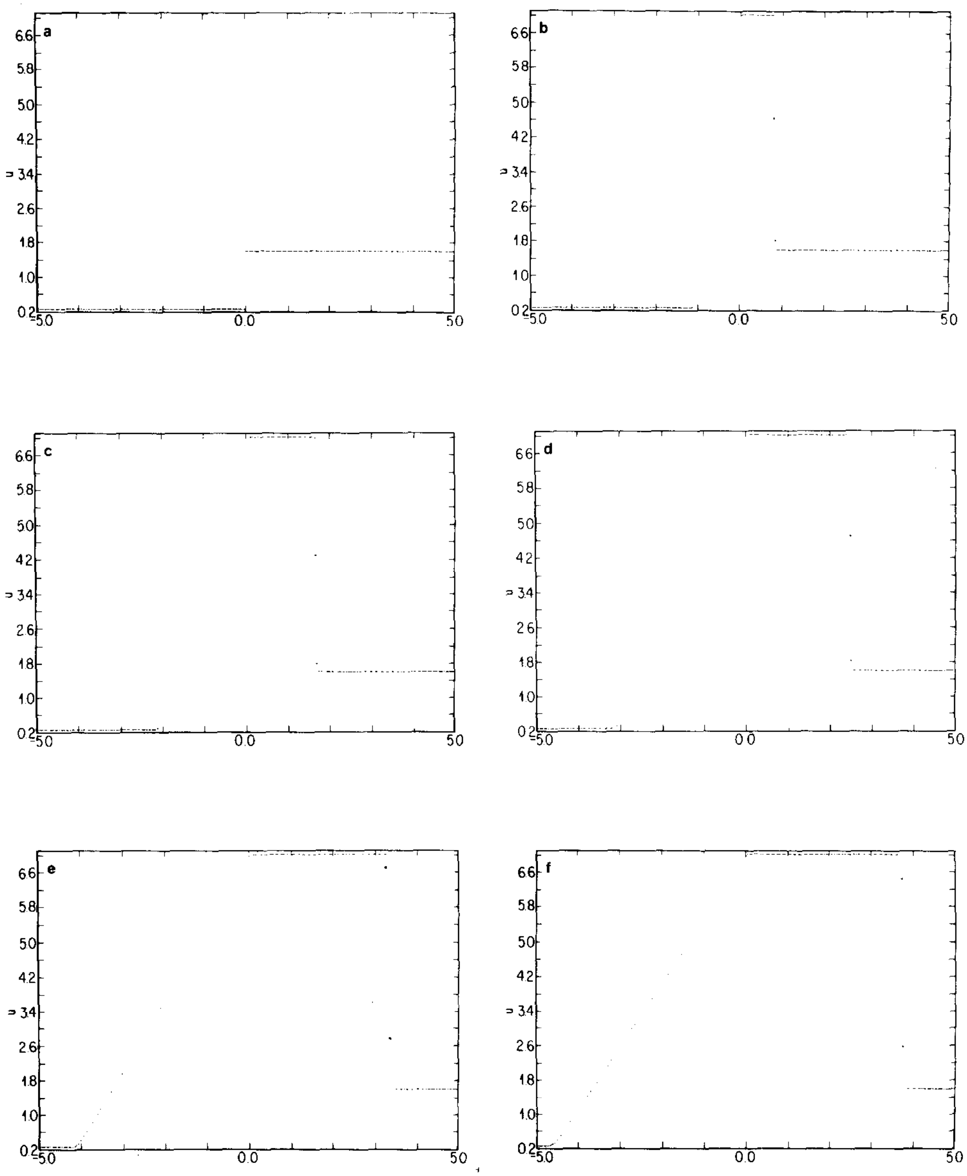

FIG. 2. Profile of the fluid velocity $u(x, t)$ as a function of x at (a) $t=0$, (b) $t=100$, (c) $t=200$, (d) $t=300$, (e) $t=400$, (f) $t=450$ sec. The distance x is in km.

step technique, and calculate first for a time step $\Delta t/2$ (the star denotes the value at half time step):

$$U_+^* = U_+\left(x - v_+ \frac{\Delta t}{2}\right), \tag{13}$$

$$U_-^* = U_-\left(x - v_- \frac{\Delta t}{2}\right). \tag{14}$$

We use U_+^* and U_-^* to calculate u^* and c^* from Eq. (11) and (12), and v_+^* and v_-^* from Eqs. (9) and (10). We use these values to advance U_+ and U_- for a full time step by calculating $U_+(x - v_+^* \Delta t)$ and $U_-(x - v_-^* \Delta t)$. The different shifted values are calculated by a simple interpolation using the second degree Bernstein polynomial presented in [6, 7].

In Ref. [5], the following parameters were used: $\Delta t = 0.2$ sec and $\Delta x = 5$ meters (initially). In the present calculation we use $\Delta t = 1.2$ sec and $\Delta x = 20$ m. The time history is shown in Fig. (1) at the same time as in Ref. [5]. The curves are smooth, and only about two points are appearing on the vertical line of the shock (the dimension of these points is slightly exagerated to make them visible). The position of the rarefaction waves agrees very well with the theory. The level of the constant state behind the shock is 4.87 m, for a theoretical value of 4.71 m (Eq. (5b)). The shock front however appears to be propagating at a velocity of 8.6 m/sec, slightly lower than the theoretical value of 10.7 m/sec (about 20% slower). This is a minor inconvenience if we take into consideration that the algorithm is absolutely stable and the otherwise very good agreement of the other parameters with the predicted theoretical values. Only 375 iterations were needed to reach the maximum time of 450 sec in Fig. 1.

For the calculation of the fluid velocity $u(x, t)$, Fig. 2 show the results at the sme time as those of Fig. 1. The value of $u(x, t)$ at the constant state behind the shock is 7.02 m/sec, for a theoretical value of 7.24 m/sec [1]. The agreement is quite good.

Finally, we developed a second algorithm for the direct solution of Eqs. (1), (2), without using the Riemann invariants. This second algorithm uses a method of fractional step similar to what is described in [11]. To advance the equations in time for a time step Δt, we first integrate for a half time step $\Delta t/2$ the equations

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = 0, \tag{15}$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \tag{16}$$

the results are used to advance the following equations for a full time step:

$$\frac{\partial h}{\partial t} + h \frac{\partial u}{\partial x} = 0, \tag{17}$$

$$\frac{\partial u}{\partial t} + g \frac{\partial h}{\partial x} = 0. \tag{18}$$

Then Eqs. (15), (16) are advanced again for a half time step $\Delta t/2$. The integration of Eqs. (15), (16) is effected by calculating h^* and u^* by shifting the value as indicated in Eqs. (13), (14). The integration of Eqs. (17) and (18) for a full time step is effected using the following shift operator:

$$h(x, t) = \tfrac{1}{2}\,[h(x+wt) + h(x-wt) - \beta u(x+wt) + \beta u(x-wt)], \tag{19}$$

$$u(x, t) = \tfrac{1}{2}\,[u(x+wt) + u(x-wt) - h(x+wt)/\beta + h(x-wt)/\beta], \tag{20}$$

where

$$w = \sqrt{gh^*}, \qquad \beta = \sqrt{h^*/g}. \tag{21}$$

Again all the shifts are effected using the same algorithm as in Ref. [6, 7]. The results are identic to those presented in Figs. 1 and 2. However with the present method, a smaller time-step and a finer grid was needed to obtain results of the same quality of those presented in Figs. 1, 2 (typically $\Delta t = 0.2$ and $\Delta x = 5$ m in the present method).

To conclude, a simple algorithm based on a shape preserving spline using a second degree Bernstein polynomial [6, 7] has been used to solve numerically the shallow water equations. The codes developed are stable. With the present parameters, the present method appears also computationally fast. It shares the important advantage of the Random Choice Method for the shock wave: absence of overshooting or undershooting at the shock discontinuity.

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